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A NEW APPROACH TO CHECK THE FREE BOUNDARY OF SINGLE FACTOR INTEREST RATE PUT OPTION

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Abstract

The application of Green's theorem to free boundary problems in option pricing leads to a new metric to measure numerical errors. Free boundaries for a variety of interest rate models are computed more accurately through minimization of our metric.

1. Introduction. The value of the free boundary at a particular time before the expiry of the option is the underlying asset value at which an American option ceases to exist. The basis of the American option pricing methodology is the location of the free boundary. Thus numerical errors in the computation of the free boundary affect the valuation of options. The paucity of analytical solutions for free boundary problems makes the problem of error measurement difficult.

A possible scheme is to assume that the option price has been calculated, and use this option price as the basis to locate the free boundary. This approach serves two purposes. First it indicates whether the numerical scheme is stable; secondly it tells us the nature and shape of the free boundary. To date only Courtadon (1982) has used option prices as the basis to locate the free boundary. Courtadon's approach was, however, very simple in that he used linear interpolation to track the free boundary. In this paper we use Green's theorem in conjunction with the Box Method to locate the free boundary. This paper represents the first attempt in the financial literature to track the free boundary in this manner.

In Section 2 we set up the American option pricing problem. In Section 3 we locate the free boundary. Section 4 compares the free boundaries of American put options based on the Vasicek model ($\gamma = 0$), CIR model ($\gamma = 0.5$) and Brennan and Schwartz model ($\gamma = 1$). Section 5 contains a summary and conclusion.

2. The American Put Option Problem

The basic starting equation is:

$$\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2\varepsilon}{\partial r^2}+k(\theta-r)\frac{\partial\varepsilon}{\partial r}-r\varepsilon=\frac{\partial\varepsilon}{\partial\tau} \qquad (1)$$

where the dynamics of the spot rate r are given by

$dr=k(\theta-r)dt+r^{\gamma}\sigma dz$

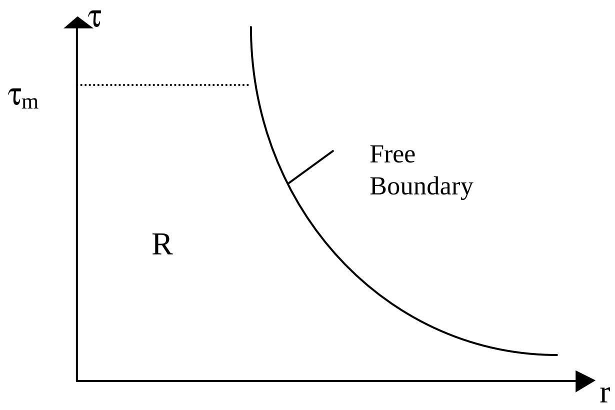
where dt is the time increment and z is a Brownian motion,

$\varepsilon = P\left(r_t, t, T^*, T\right)+B\left(r_t, t, T^*\right)=$ Put on bond + Bond Price .

Further at the free boundary the two following boundary conditions hold:

$$\frac{\partial\varepsilon\left(s(\tau),\tau\right)}{\partial r}=0 \qquad (B1)$$

$$\varepsilon\left(s(\tau),\tau\right) = E \text{ (Exercise Price)} \qquad (B2)$$



In the diagram above the curve $r = s(\tau)$ is the free boundary. We integrate equation (1) in the region R bounded by the free boundary . In particular along the time axis we integrate from $0 \to \tau_m$,the largest time at which we compute the free boundary, at time increment $m\Delta t$. We integrate from $0 \to s(\tau)$,the critical interest rate at time τ, along the interest rate axis.

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau + \iint_R k\theta \frac{\partial \varepsilon}{\partial r} dr d\tau - \iint_R rk \frac{\partial \varepsilon}{\partial r} dr d\tau$$

$$- \iint_R r\varepsilon dr d\tau = \iint_R \frac{\partial \varepsilon}{\partial \tau} dr d\tau \qquad (2)$$

We now integrate and simplify each component of the above equation, starting with the first component $\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$. In particular with the first component, we consider four distinct cases, first $\gamma = 0$, second $\gamma = \frac{1}{2}$, third $\gamma = 1$, and for γ between 0 and 1 excluding the previous values of γ.

First consider the case for $\gamma = 0$

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \iint_R \frac{\sigma^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Now integrating by parts and incorporating boundary condition B1 gives:

$$\int_0^{s(\tau)} \frac{\partial^2 \varepsilon}{\partial r^2} dr = -\frac{\partial \varepsilon(0,\tau)}{\partial r}$$

Further integration of the above expression with respect to time gives:

$$\int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\frac{\sigma^2}{2} \int_0^{\tau_m} \frac{\partial \varepsilon(0,\tau)}{\partial r} d\tau$$

Second consider the integral for $\gamma = \frac{1}{2}$

$$\iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \iint_R \frac{\sigma^2 r}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Integrating the integral $\int_0^{s(\tau)} \frac{\sigma^2 r}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr$ by parts and inserting the second boundary condition (B2) gives:

$$\int_0^{s(\tau)} \frac{\sigma^2}{2} r \frac{\partial^2 \varepsilon}{\partial r^2} dr = \frac{\sigma^2}{2} \left[\varepsilon(0,\tau) - E \right]$$

Further integrating the above expression with respect to time gives us:

$$\int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = -\frac{\sigma^2}{2} E(m\Delta t) + \frac{\sigma^2}{2} \int_0^{\tau_m} \varepsilon(0,\tau) d\tau$$

Thirdly consider the integral for $\gamma = 1$

$$\iint_R \frac{\sigma^2}{2} r^2 \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau = \int_0^{\tau_m} \int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr d\tau$$

Integrating the component $\int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr$ by parts and incorporating the boundary condition B2 gives:

$$\int_0^{s(\tau)} \frac{\sigma^2 r^2}{2} \frac{\partial^2 \varepsilon}{\partial r^2} dr = -\sigma^2 \int_0^{s(\tau)} r \frac{\partial \varepsilon}{\partial r} dr = -\sigma^2 s(\tau) E + \sigma^2 \int_0^{s(\tau)} \varepsilon(r,\tau) dr$$

Once again further integrating the above expression with respect to time gives us:

$$\int_0^{\tau_m}\int_0^{s(\tau)}\frac{\sigma^2 r^2}{2}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = -\sigma^2 E\int_0^{\tau_m}s(\tau)d\tau + \sigma^2\int_0^{\tau_m}\int_0^{s(\tau)}\varepsilon(r,\tau)drd\tau$$

Now for the general case of γ between 0 and 1 and excluding the particular values mentioned above, we have by integrating by parts and by incorporating the boundary condition B1:

$$\iint_R \frac{\sigma^2}{2}r^{2\gamma}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = \int_0^{\tau_m}\int_0^{s(\tau)}\frac{\sigma^2 r^{2\gamma}}{2}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = -\gamma\sigma^2\int_0^{\tau_m}\int_0^{s(\tau)}r^{2\gamma-1}\frac{\partial\varepsilon}{\partial r}drd\tau$$

We now further integrate the component $\int_0^{s(\tau)}r^{2\gamma-1}\frac{\partial\varepsilon}{\partial r}dr$ by parts and insert boundary condition (B1) to give:

$$\int_0^{\tau}r^{2\gamma-1}\frac{\partial\varepsilon}{\partial r}dr = Es(\tau)^{2\gamma-1} - (2\gamma-1)\int_0^{s(\tau)}r^{2\gamma-2}\varepsilon(r,\tau)dr$$

Thus in the expanded form, the double integral for the general case is:

$$\int_0^{\tau_m}\int_0^{s(\tau)}\frac{\sigma^2 r^{2\gamma}}{2}\frac{\partial^2\varepsilon}{\partial r^2}drd\tau = \sigma^2\gamma(2\gamma-1)\int_0^{\tau_m}\int_0^{s(\tau)}r^{2\gamma-2}\varepsilon(r,\tau)drd\tau - \sigma^2\gamma E\int_0^{\tau_m}s(\tau)^{2\gamma-1}d\tau$$

Note that the above expression also holds for $\gamma = 1$. Thus summarizing all the possible expressions:

$$LHS_0 = \iint_R \frac{\sigma^2}{2} r^{2\gamma} \frac{\partial^2 \varepsilon}{\partial r^2} drd\tau = \begin{cases} -\frac{\sigma^2}{2}\int_0^{\tau_m} \frac{\partial \varepsilon(0,\tau)dt}{\partial r}, \gamma = 0 \\ -\frac{\sigma^2}{2} E(m\Delta t) + \frac{\sigma^2}{2}\int_0^{\tau_m} \varepsilon(0,\tau)d\tau, \gamma = \frac{1}{2} \\ \sigma^2\gamma(2\gamma-1)\int_0^{\tau_m}\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r,\tau)drd\tau - \sigma^2\gamma E\int_0^{\tau_m} s(\tau)^{2\gamma-1}d\tau, \gamma \neq 0, \gamma \neq \frac{1}{2} \end{cases}$$

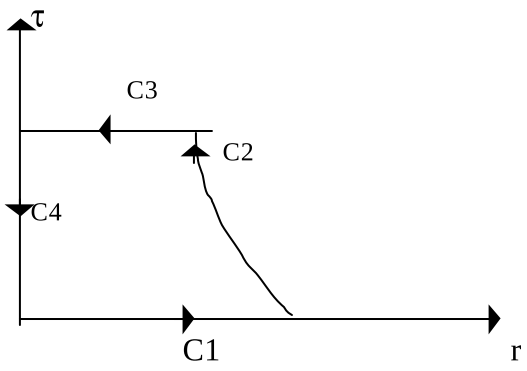
Now integrating the second component of equation (2) and inserting boundary condition B2 gives:

$$k\theta \iint_R \frac{\partial \varepsilon}{\partial r} drd\tau = k\theta \int_0^{\tau_m}\int_0^{s(\tau)} \frac{\partial \varepsilon}{\partial r} drd\tau = k\theta E(m\Delta t) - k\theta \int_o^{\tau_m} \varepsilon(0,\tau)d\tau$$

Finally integrating the third component of equation (2) by parts and inserting boundary condition B2 gives us:

$$-k\iint_R r\frac{\partial \varepsilon}{\partial r} drd\tau = -Ek\int_0^{\tau_m} s(\tau)d\tau + k\int_0^{\tau_m}\int_0^{s(\tau)} \varepsilon(r,\tau)drd\tau$$

We now consider the term on the right hand side of the original equation. The figure below indicates the path of integration followed (C1,C2,C3,C4).



Applying Green's theorem gives us:

$$\iint_R \frac{\partial \varepsilon(r,\tau)}{\partial \tau} = -\oint_{C1+C2+C3+C4} \varepsilon(r,\tau)dr$$

We now evaluate each of the components of the above integral separately:

$\int_{C4} \varepsilon(r,\tau)dr = 0$ as we are moving along the time axis only where the interest rate is constant.

$$\int_{C1} \varepsilon(r,\tau)dr = \int_0^{s(0)} \varepsilon(r,0)dr$$

We note that C2 is the free boundary and that from boundary condition B2 along C2 $\varepsilon(r,\tau) = E$. Hence:

$$\int_{C2} \varepsilon(r,\tau)dr = \int_{\tau=0}^{\tau_m} E\frac{dr}{d\tau}d\tau = E[s(\tau_m) - s(0)]$$

$$\int_{C3} \varepsilon(r,\tau)dr = -\int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

Collecting all the terms on the right hand side gives us:

$$\iint_R \frac{\partial \varepsilon}{\partial \tau} drd\tau = \int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr + E[s(0) - s(\tau_m)] - \int_0^{s(0)} \varepsilon(r,0)dr$$

Collecting and rearranging the terms both on the left-hand side and the right hand side of equation (2) gives us:

$$LHS_0 + LHS_1 + LHS_2 + LHS_3 + LHS_4 + LHS_5 + LHS_6 = RHS_0 + RHS_1 \quad (3)$$

where:

$$LHS_1 = k\theta E(m\Delta t)$$

$$LHS_2 = -k\theta \int_0^{\tau_m} \varepsilon(0,\tau)d\tau$$

$$LHS_3 = \int_0^{\tau_m} \int_0^{s(t)} (k - r)\varepsilon(r,\tau)drd\tau$$

$$LHS_4 = -kE \int_0^{s(\tau)} s(\tau)d\tau$$

$$LHS_5 = -Es(0)$$

$$LHS_6 = \int_0^{s(0)} \varepsilon(r,0)dr$$

$$RHS_0 = \int_0^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

$$RHS_1 = -Es(\tau_m)$$

Observing equation (3) we see that at any general time step τ_m there is no analytical solution for $s(\tau_m)$. In the appendix we use numerical integration to solve equation (3).

3. Locating the Free Boundary

At the maturity date of the contingent claim we define the following function discretized at interest rate point r_k:

$$\phi_k = E - B(r_k)$$

If we let r_{k-2}, r_{k-1}, r_k and r_{k+1} be interest rate points and ϕ_{k-2}, ϕ_{k-1}, ϕ_k and ϕ_{k+1} be values of the above function at these interest rates. Then, we can derive the following polynomial:

$$A(r) = \sum_{l=k-2}^{k+1} \phi_l L_l(r)$$

where

$$L_l(r) = \prod_{l=k-2, l \neq k}^{k+1} \frac{r - r_l}{r_k - r_l}$$

with the following property:

$$\phi_l = A(r_l) \qquad l = k-2,\ k-1,\ k,\ k+1$$

We now use Newton-Raphson iteration, to derive the critical interest rate $s(0)$ at expiry date of the put option.

$$s(0) = r - \frac{A(r)}{\frac{d}{ds}\left[A(r)\right]}$$

At general time step $m\Delta t$, the free boundary is located by solving for the zero of the function:

$$\phi = \phi_{LHS} - \phi_{RHS}$$

where:

$$\phi_{LHS} = LHS_0 + LHS_1 + LHS_2 + LHS_3 + LHS_4 + LHS_5 + LHS_6$$

$$\phi_{RHS} = RHS_0 + RHS_1$$

Numerical experimentation indicates that Newton-Raphson is not suitable except at the maturity date of the option. Thus at general time step $m\Delta t$, we start with a value of $s(m\Delta t)$ which by examination of the grid at this time step is known to be lower than the actual value of $s(m\Delta t)$. To estimate a more accurate $s(m\Delta t)$ we iterate upwards at interest rate steps of $\Delta r/20$ until the following criterion is met:

$$\left|\phi_{LHS} - \phi_{RHS}\right| = \left|10^{-5}\right|$$

Once this criterion is met we move to the next time step to calculate $s((m+1)\Delta t)$ and so on until we reach to end of the grid at time step $M\Delta t$.

We now investigate the nature of the free boundary of American put options based on widely used single factor term structure models. In particular we consider the Vasicek model $(\gamma = 0)$, CIR model $\left(\gamma = \frac{1}{2}\right)$ and Brennan-Schwartz model $(\gamma = 1)$. All three models are of course enclosed by the more general CKLS model. We investigate the free boundary both for short expiry and long expiry put options. The short expiry options are based on bonds with 5-year maturity bond and expiry of 1 year. The longer expiry put options are based on 10-year bonds and expiry of 5 years. The bonds are zero coupon and have a face value of 100.00. The parameters take the following values: $\sigma = 0.5$, $k = 0.1$, $\theta = 0.08$. On the grid the interest rate spacing is $\Delta r = 0.05$ and the time intervals of $\Delta t = 0.002$.

4. Analysis

We plot the free boundaries for $\gamma = 0$ (Vasicek), $\gamma = 0.5$ (CIR) and $\gamma = 1$ (Brennan-Schwartz). For each γ value two sets of free boundaries are plotted at different exercise prices. The terms to expiry of the put options are either 1 year or 5 years. The 1 year put options are priced on a 5-year bond during the last year before it matures. The 5 year put options are priced on a 10-year bond during the last 5-year before it matures. All the free boundaries are plotted backwards in time that is, we start plotting from the expiry date of the options to current date at which the put option is written.

For $\gamma = 0$, 1 year put option (Figure 1) the critical interest rate increases rapidly. However, as the current date of the option approaches, the critical interest rate increases asymptotically; such that by the current date the free boundary is almost flat. For 5 year options (Figure 2), the critical interest rate increases rapidly close to the expiry date of the option as with $\gamma = 0$. Although the free boundary is almost flat by the current date careful examination of the graph indicates that critical interest rate actually start to decrease as the current date of the put option approaches. This is in contrast to the free boundary of the one-year option. Note that critical bond prices are monotonous in any case.

For $\gamma = 0.5$, 1 year put option (Figure 3), the free boundary evolves in the same way as for $\gamma = 0$. For 5 year put option (Figure 4), the free boundary increases close to the maturity date of the option. However as the current date of the option approaches, the critical interests show a noticeable decline. The end result is that for a 5 year put option, the free boundary initially increases and then declines asymptotically.

For $\gamma = 1$, 1 year put option (Figure 5), the free boundary initially increases close to the maturity date and the declines as the current date approaches. This is in contrast to the behavior of free boundaries for $\gamma = 0$ and $\gamma = 0.5$. For 5 year put option (Figure 6), the critical interest initially increases, but then quickly declines. Although the free boundary in this case shows the same overall behaviour as the free boundaries for $\gamma = 0$ and $\gamma = 0.5$, there is in this case two distinct observable differences. First the critical interest rate starts to decline much closer to the maturity date than for $\gamma = 0$ and $\gamma = 0.5$. Secondly the rate of decline i.e. the downward steepness of the free boundaries is greater than for $\gamma = 0$ and $\gamma = 0.5$.

Figure1 - Figure 6 all exhibit discontinuities at the expiry date and close to the expiry date of the options. This is due to an inconsistency in our model at maturity because at maturity we assume $\frac{\partial \varepsilon(s(0),0)}{\partial r} = 0$, when $\frac{\partial \varepsilon(P(s(0)),0)}{\partial r} \neq 0$. Further, although none of the free boundaries show any discontinuities except at and near the expiry date, the free boundaries nonetheless do exhibit small oscillations. This oscillation is due to the approximations we have made in setting up the grid and secondly the small errors in the critical interest rate from previous time periods feeding through to the critical interest rate at the current time period.

5. Conclusion

Since Courtadon (1982) used a linear interpolation approach to track the free boundary of interest rate contingent claims, no further research has been done to extend this work. In this paper we have provided a new method to check and track the free boundary. We have applied this new approach to check the free boundary of short dated and long dated American put options based on widely used one factor interest rate models. Our findings show how the shape of the free boundary varies from model to model and with the term to expiry of the options. Generally, we observe that the free boundary increases asymptotically towards the current date, such that by the current date the free boundary is almost flat or slightly declining.

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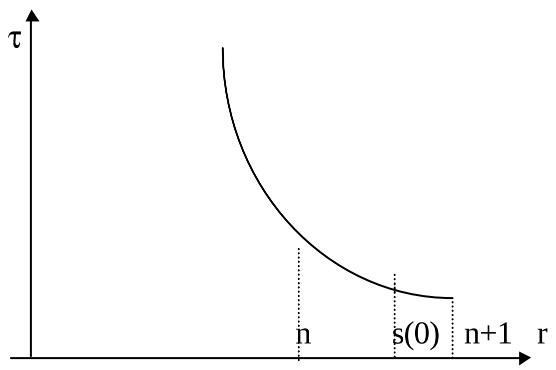
Appendix. Discretization of the Integral Equation

Each of the single integral is discretized using the implicit trapezium rule. We start by discretizing the simplest integrals first:

$$\int_0^{\tau_m} \varepsilon(0,t)d\tau \approx \Delta t\left[\frac{1}{2}\varepsilon(0,0) + \varepsilon(0,\Delta t) + \varepsilon(0,2\Delta t) + \ldots + \varepsilon(0,(m-1)\Delta t) + \frac{1}{2}\varepsilon(0,m\Delta t)\right]$$

$$\int_0^{\tau_m} s(\tau)d\tau = \Delta t\left[\frac{1}{2}s(0) + s(\Delta t) + s(2\Delta t) + \ldots + s((m-1)\Delta t) + \frac{1}{2}s(m\Delta t)\right]$$

$$\int_0^{\tau_m} s(\tau)^{2\gamma-1}d\tau = \Delta t\left[\frac{1}{2}s(0)^{2\gamma-1} + s(\Delta t)^{2\gamma-1} + s(2\Delta t)^{2\gamma-1} + \ldots + s((m-1)\Delta t)^{2\gamma-1} + \frac{1}{2}s(m\Delta t)^{2\gamma-1}\right]$$



At time 0, we separate the integral $\int_{0}^{s(0)} \varepsilon(r,0)dr$ into two components as follows with

$n_0 Dr < s(0) < (n_0 + 1)Dr$

$$\int_{0}^{s(0)} \varepsilon(r,0)dr = \int_{0}^{n_0 Dr} \varepsilon(r,0)dr + \int_{n_0 Dr}^{s(0)} \varepsilon(r,0)dr$$

We discretize each of the two integrals using the implicit trapezium rule as follows:

$$\int_{0}^{n_0 Dr} \varepsilon(r,0)dr \approx Dr\left[\frac{1}{2}\varepsilon(0,0) + \varepsilon(Dr,0) + \varepsilon(2Dr,0) + \ldots.\varepsilon\big((n_0 - 1)Dr,0\big) + \frac{1}{2}\varepsilon\big(n_0 Dr,0\big)\right]$$

$$\int_{n_0 \Delta r}^{s(0)} \varepsilon(r,0)dr \approx \frac{\big(s(0) - n_0 \Delta r\big)}{2}\big[\varepsilon\big(n_0 \Delta r,0\big) + E\big]$$

Combining the above two discretizations gives us:

$$\int_{0}^{s(0)} \varepsilon(r,0)dr = \Delta r\left[\frac{1}{2}\varepsilon(0,0) + \varepsilon(\Delta r,0) + \varepsilon(2\Delta r,0) + \ldots + \varepsilon\big((n_0 - 1)\Delta r,0\big) + \frac{1}{2}\varepsilon\big(n_0 \Delta r,0\big)\right]$$
$$+ \frac{\big(s(0) - n_0 \Delta r\big)}{2}\big[\varepsilon\big(n_0 \Delta r,0\big) + E\big]$$

At time τ_m, as at time 0, we separate the integral $\int_{0}^{s(\tau_m)} \varepsilon(r,\tau_m)dr$ into two components as follows with $n_m \Delta r < s(\tau_m) < (n_m + 1)\Delta r$

$$\int_{0}^{s(\tau_m)} \varepsilon(r,\tau_m)dr = \int_{0}^{n_m\Delta r} \varepsilon(r,\tau_m)dr + \int_{n_m\Delta r}^{s(\tau_m)} \varepsilon(r,\tau_m)dr$$

We discretize each of the two integrals using the implicit trapezium rule as follows:

$$\int_{0}^{n_m\Delta r} \varepsilon(r,\tau_m)dr \approx \Delta r\left[\frac{1}{2}\varepsilon(0,\tau_m) + \varepsilon(\Delta r,\tau_m) + \varepsilon(2\Delta r,\tau_m) + \ldots.\varepsilon((n_m - 1)\Delta r,\tau_m) + \frac{1}{2}\varepsilon(n_m\Delta r,\tau_m)\right]$$

$$\int_{n_m\Delta r}^{s(\tau_m)} \varepsilon(r,0)dr \approx \frac{(s(\tau_m) - n_m\Delta r)}{2}\left[\varepsilon(n_m\Delta r,0) + E\right]$$

Combining the above two discretizations gives us:

$$\int_{0}^{s(\tau_m)} \varepsilon(r,\tau_m)dr \approx \Delta r\left[\frac{1}{2}\varepsilon(0,\tau_m) + \varepsilon(\Delta r,\tau_m) + \varepsilon(2\Delta r,\tau_m) + \ldots + \varepsilon((n_m - 1)\Delta r,\tau_m) + \frac{1}{2}\varepsilon(n_m\Delta r,\tau_m)\right]$$
$$+\frac{(s(\tau_m) - n_m\Delta r)}{2}\left[\varepsilon(n_m\Delta r,0) + E\right]$$

For the integral $-\frac{\sigma^2}{2}\int_{0}^{\tau_m} \frac{\partial\varepsilon(0,\tau)}{\partial r}d\tau$ we first discretize $\frac{\partial\varepsilon(0,\tau)}{\partial r}$ using the forward difference approximation such that:

$$\frac{\partial\varepsilon(0,\tau)}{\partial r} \approx \frac{\varepsilon(\Delta r,\tau) - \varepsilon(0,\tau)}{\Delta r}$$

Substituting the above expression into the original integral gives us:

$$-\frac{\sigma^2}{2}\int_0^{\tau_m}\frac{\partial\varepsilon(0,\tau)}{\partial\tau}d\tau=-\frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(\Delta r,\tau)d\tau+\frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau$$

Discretizing each of the components of the above equation gives gives us:

$$\frac{-\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau\approx\frac{-\sigma^2\Delta t}{2\Delta r}\left[\frac{1}{2}\varepsilon(0,0)+\varepsilon(0,\Delta t)+\varepsilon(0,2\Delta t)+\ldots+\varepsilon(0,(m-1)\Delta t)+\frac{1}{2}\varepsilon(0,m\Delta t)\right]$$

$$\frac{\sigma^2}{2\Delta r}\int_0^{\tau_m}\varepsilon(0,\tau)d\tau\approx\frac{\sigma^2\Delta t}{2\Delta r}\left[\frac{1}{2}\varepsilon(\Delta r,0)+\varepsilon(0,\Delta t)+\varepsilon(\Delta r,2\Delta t)+\ldots+\varepsilon(\Delta r,(m-1)\Delta t)+\frac{1}{2}\varepsilon(\Delta r,m\Delta t)\right]$$

Combining the above two discretizations gives us:

$$-\frac{\sigma^2}{2}\int_0^{\tau_m}\frac{\partial\varepsilon(0,\tau)}{\partial\tau}dt=-\frac{\sigma^2\Delta t}{2\Delta r}\left[\begin{array}{l}\frac{1}{2}\big(\varepsilon(0,0)-\varepsilon(\Delta r,0)\big)+\big(\varepsilon(0,\Delta t)-\varepsilon(\Delta r,\Delta t)\big)+\ldots\\+\big(\varepsilon(0,(m-1)\Delta t)-\varepsilon(\Delta r,(m-1)\Delta t)\big)+\frac{1}{2}\big(\varepsilon(0,m\Delta t)-\varepsilon(\Delta r,m\Delta t)\big)\end{array}\right]$$

To discretize the double integrals we first change the order of integration as follows:

$$\int_0^{\tau_m}\int_0^{s(\tau)}(k-r)\varepsilon(r,\tau)drd\tau=\int_0^{s(\tau)}\left[\int_0^{\tau_m}(k-r)\varepsilon(r,\tau)d\tau\right]dr$$

$$\int_0^{\tau_m}\int_0^{s(\tau)}r^{2\gamma-2}\varepsilon(r,\tau)drd\tau=\int_0^{s(\tau)}\left[\int_0^{\tau_m}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr$$

We now discretize the above double integrals at successive time steps

First at time period Δt :

$$\int_0^{s(\tau)}\left[\int_0^{\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}(k-r)\varepsilon(r,0)dr+\frac{\Delta t}{2}\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{\Delta t}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}r^{2\gamma-2}\varepsilon(r,0)dr+\frac{\Delta t}{2}\int_0^{s(\Delta t)}r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

At time period $2\Delta t$:

$$\int_0^{s(\tau)}\left[\int_0^{2\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}(k+-r)\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$+\frac{\Delta t}{2}\int_0^{s(2\Delta t)}(k-r)\varepsilon(r,2\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{2\Delta t}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}r^{2\gamma-2}\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

$$+\frac{\Delta t}{2}\int_0^{s(\Delta t)}r^{2\gamma-2}\varepsilon(r,2\Delta t)dr$$

At time period $m\Delta t$:

$$\int_0^{s(\tau)}\left[\int_0^{m\Delta t}(k-r)\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}(k-r)\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}(k-r)\varepsilon(r,\Delta t)dr$$

$$+\Delta t\int_0^{s(2\Delta t)}(k-r)\varepsilon(r,2\Delta t)dr+\ldots+\Delta t\int_0^{s((m-1)\Delta t)}(k-r)\varepsilon(r,(m-1)\Delta t)dr+\frac{\Delta t}{2}\int_0^{s(m\Delta t)}(k-r)\varepsilon(r,m\Delta t)dr$$

$$\int_0^{s(\tau)}\left[\int_0^{m\Delta t}r^{2\gamma-2}\varepsilon(r,\tau)d\tau\right]dr=\frac{\Delta t}{2}\int_0^{s(0)}r^{2\gamma-2}\varepsilon(r,0)dr+\Delta t\int_0^{s(\Delta t)}r^{2\gamma-2}\varepsilon(r,\Delta t)dr$$

$$+\Delta t\int_0^{s(2\Delta t)}r^{2\gamma-2}\varepsilon(r,2\Delta t)dr+\ldots\Delta t\int_0^{s((m-1)\Delta t)}r^{2\gamma-2}\varepsilon(r,(m-1)\Delta t)dr+\frac{\Delta t}{2}\int_0^{s(m\Delta t)}r^{2\gamma-2}\varepsilon(r,m\Delta t)dr$$

We note that the above integrals are similar to $\int_0^{s(\tau_m)}\varepsilon(r,\tau_m)dr$ and hence discretized as follows:

$$Y_m=\int_0^{s(m\Delta t)}(k-r)\varepsilon(r,m\Delta t)dr=\begin{bmatrix}\frac{1}{2}k\varepsilon(0,\tau_m)+(k-\Delta r)\varepsilon(\Delta r,m\Delta t)\Delta r\\ +(k-2\Delta r)\varepsilon(2\Delta r,m\Delta t)\Delta r\\ +\ldots+(k-(n_m-1)\Delta r)\varepsilon((n_m-1)\Delta r,m\Delta t)\Delta r\\ +\frac{1}{2}(k-m\Delta r)\varepsilon(n_m\Delta r,m\Delta t)\Delta r+\\ \left(\frac{s(m\Delta t)-m\Delta r}{2}\right)\times\\ ((k-m\Delta r)\varepsilon(n_m\Delta r,m\Delta t)+(k-s(m\Delta t))E)\end{bmatrix}$$

$$Z_m = \int_0^{s(m\Delta t)} r^{2\gamma-2}\varepsilon(r,m\Delta t)dr = \begin{bmatrix} (\Delta r)^{2\gamma-2}\varepsilon(\Delta r,\tau_m)\Delta r + (2\Delta r)^{2\gamma-2}\varepsilon(2\Delta r,m\Delta t)\Delta r \\ +\ldots+((n_m-1)\Delta r)^{2\gamma-2}\varepsilon((n_m-1)\Delta r,m\Delta t)\Delta r \\ +\frac{1}{2}(m\Delta r)^{2\gamma-2}\varepsilon(n_m\Delta r,m\Delta t)\Delta r + \\ \left(\frac{s(m\Delta t)-m\Delta r}{2}\right)\times \\ \left((m\Delta r)^{2\gamma-2}\varepsilon(n_m\Delta r,m\Delta t)+(s(m\Delta t))^{2\gamma-2}E\right) \end{bmatrix}$$

Thus summarizing both the above double integrals,we have:

$$\int_0^{\tau_m}\int_0^{s(\tau)}(k+\lambda-r)\varepsilon(r,\tau)\,drd\tau = \frac{\Delta t}{2}Y_1 + \Delta tY_2 + \ldots + \Delta tY_{m-1} + \frac{\Delta t}{2}Y_m$$

$$\int_0^{\tau_m}\int_0^{s(\tau)} r^{2\gamma-2}\varepsilon(r,\tau)\,drd\tau = \frac{\Delta t}{2}Z_1 + \Delta tZ_2 + \ldots + \Delta tZ_{m-1} + \frac{\Delta t}{2}Z_m$$

Figure1: Vasicek model, 5 year bond, 1 year put option

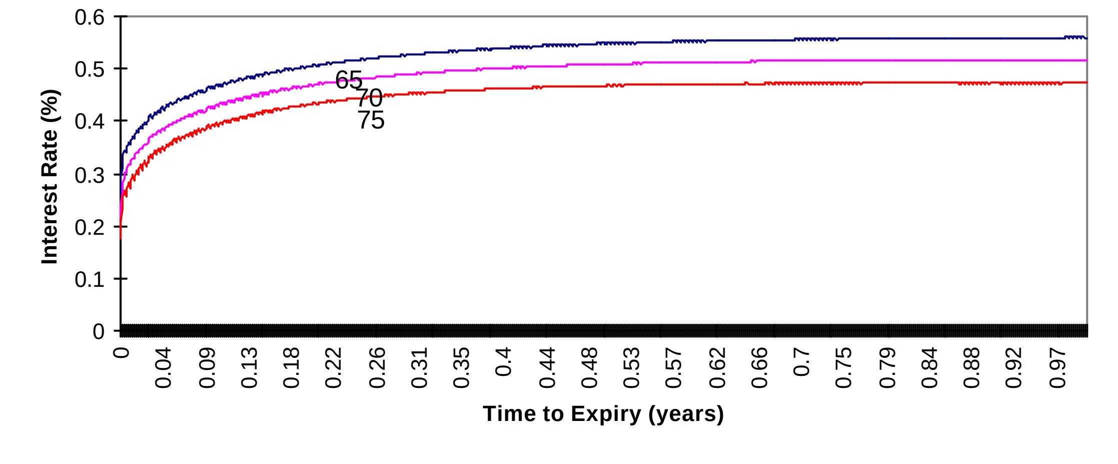


Figure 2: Vasicek model, 10 year bond, 5 year put option

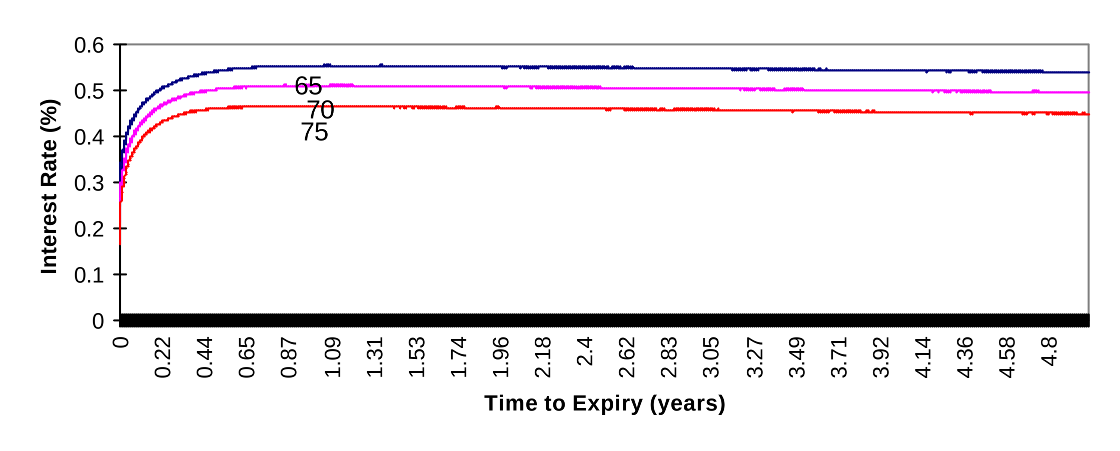


Figure3: CIR Model, 5 year bond, 1 year put option

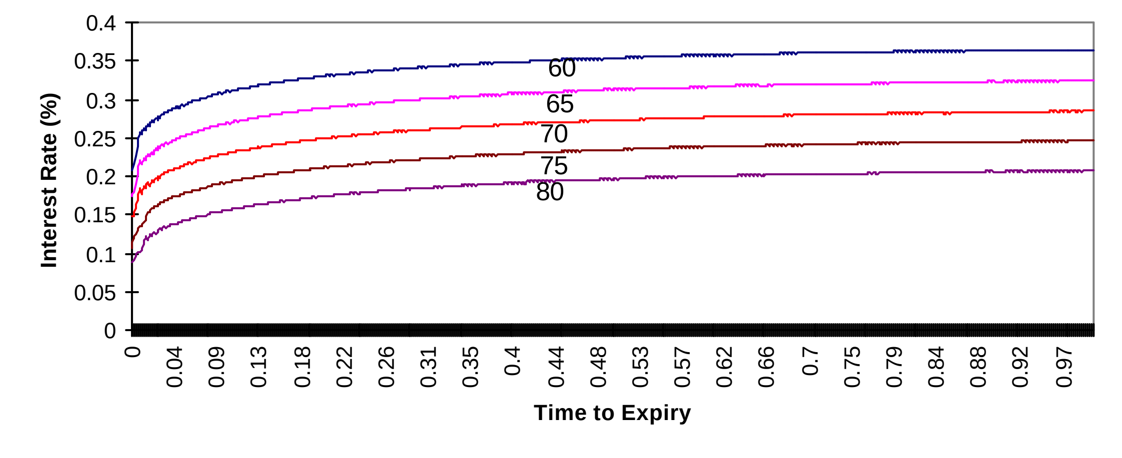


Figure 4: CIR model, 10 year bond, 5 year put option

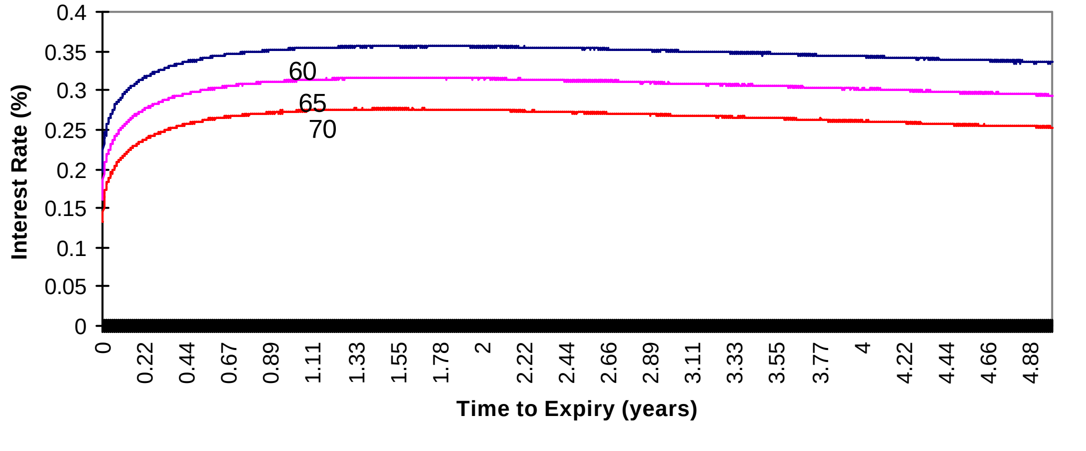


Figure 5: Brennan-Schwartz model, 5 year bond, 1year put option

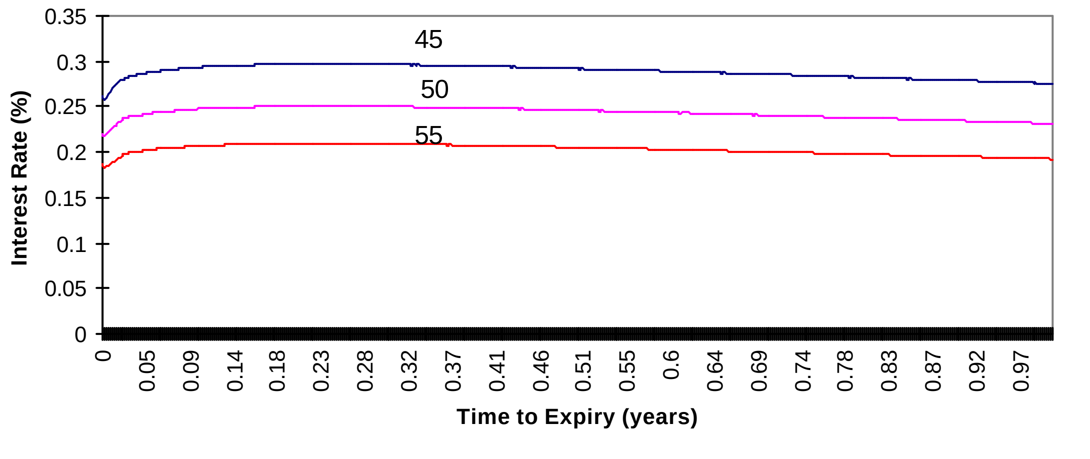
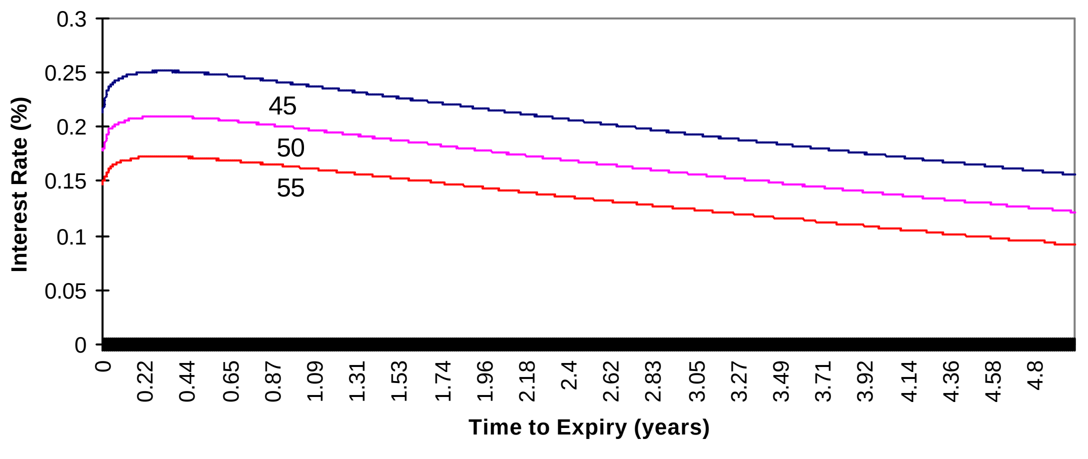


Figure 6: Brennan-Schwartz model, 10 year bond, 5 year put option



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